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Summing over Feynman histories by functional contour integration

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Abstract. We show how complex paths can be consistently introduced into sums over Feynman histories by using the notion of functional contour integration. For a k-dimensional system specified by a potential with suitable analyticity properties, each coordinate axis is replaced by a copy of the complex plane, and at each instant of time a contour is chosen in each plane. This map from the time axis into the set of complex contours defines a functional contour. The family of contours labelled by time generates a $(k+1)$-dimensional submanifold of the $(2k+1)$-dimensional space defined by the cartesian product of the time axis and the coordinate planes. The complex Feynman paths lie on this submanifold. The convergence problems encountered in previous proposals for complex path integrals are avoided by the requirement that each contour is asymptotically pinched to the real coordinate axis. An application of this idea to systems described by absorptive potentials yields a simple derivation of the correct WKB result in terms of a complex path that extremalises the action. The method can also be applied to spherically symmetric potentials by using a partial wave expansion and restricting the contours appropriately.

1. Introduction

Feynman's representation of quantum mechanics as a sum over histories (Feynman and Hibbs 1965) was first formulated to deal with conservative systems having Hermitian Hamiltonians, but it has also proved useful in the treatment of dissipative systems phenomenologically described by complex potentials. The most prominent examples are scattering in a nuclear optical-model potential (McLaughlin 1972a, Koeling and Malfliet 1975, Knoll and Schaeffer 1976) and propagation of laser beams in absorptive media (Kogelnik 1965, Arnaud 1976). The path integral treatment of these problems can be given in terms of sums over conventional real trajectories (McLaughlin 1972a, Exner 1982, Exner and Kolerow 1982, Wright 1984); however, the classical action now possesses an imaginary part so that it is not possible to obtain the usual semiclassical results associated with the trajectory that extremalises the action. It is possible to find a trajectory for which the real part of the action is stationary, but this path does not necessarily dominate the integral. The imaginary part of the action produces a damping which varies with the path; consequently it is necessary to determine the set of real paths with minimum damping in order to find an approximation

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to the integral. A number of authors (McLaughlin 1972a, Koeling and Malfliet 1975, Knoll and Schaeffer 1976) have attempted to avoid this problem by generalising the Feynman integral to include all complex paths joining the prescribed initial and final points. These theories are plagued with extreme divergence difficulties, so that integration over complex paths is even less well understood than integration over real paths. The objective of the present paper is to show that complex paths can be consistently introduced into the theory of Feynman path integrals by means of the notion of functional contour integration. We stress that, in common with ordinary contour integration, this technique simply provides a new representation of the original theory which is exactly equivalent to the conventional formulation in terms of real trajectories. The advantage of the new representation is that a suitable choice of the functional contour yields a simple calculation of the semiclassical limit. An illustrative application of this procedure can be found in a previous paper (Garrison and Wright 1985). In the present paper we do not present any new results for propagators; instead we are concerned with justifying results previously obtained by *ad hoc* methods.

In § 2 we define the notion of functional contour integration for k-dimensional systems described by potentials which are analytic except for fixed singularities, i.e. the locations of the singularities in any coordinate are independent of the values of the other coordinates. Under these circumstances we show in § 3 that the correct WKB result for an absorptive potential can be obtained by a simple stationary action calculation. In § 4 we extend the method to include problems involving spherically symmetric potentials. This is done by using the well known partial wave expansions (Edwards and Gulyaev 1964, Peak and Inomata 1969, Langguth and Inomata 1979) and deforming the radial path integral by means of contours which lie in the right half of the complex radius plane. In § 5 we make a few remarks on the problem of calculating barrier penetration factors with the path integral method, and in § 6 we give a summary and conclusions.

2. Functional contour integrals

For definiteness we consider a quantum mechanical system with a k-dimensional configuration space and a complex potential $V(x)$ given by

$$V(\boldsymbol{x}) = V_{\mathrm{S}}(\boldsymbol{x}) - \mathrm{i}\hbar V_{\mathrm{A}}(\boldsymbol{x}) \tag{2.1}$$

where $\boldsymbol{x} = (x_1, \ldots, x_k)$, the functions V_{S} and V_{A} are real and V_{A} is non-negative. The explicit factor of $\hbar$ is associated with the absorptive part of the potential in order to conform with the usual conventions (Guillod and Huguenin 1984). We restrict the potential by the assumption that it is jointly analytic in the (cartesian) coordinates $x_1, \ldots, x_k$. That is, the function originally defined on the real configuration space $\boldsymbol{R}^k$ can be extended into an analytic function defined on the complex configuration space $\boldsymbol{C}^k$. Thus each coordinate axis is replaced by a (complex) coordinate plane, and $x_\alpha \to z_\alpha$, $\alpha = 1, \ldots, k$. Furthermore, the location of any singularity, for example a pole or branch point, in a given variable z_α is required to be independent of the values of the other variables. We refer to such singularities as fixed. Potentials with non-fixed singularities would not be generally amenable to our technique; however, in some cases, such as a spherically symmetric potential, it is possible to recast the problem so that it can be treated by this method. For one-dimensional problems the fixed-singularity condition is trivially satisfied. An example of a multidimensional problem

with this property is provided by an anharmonic molecular model in which the interaction between the vibrational modes is given by a finite-order polynomial in the variables $x_1, \ldots, x_k$. In this case the singularities in the analytically continued potential come entirely from the terms involving only one of the coordinates at a time.

The Hamiltonian for the k-dimensional system is

$$H = \sum_{\alpha=1}^{k} \tfrac{1}{2}p_\alpha^2 + V(\boldsymbol{x})$$

where units have been chosen so that the masses are unity. The condition that V_A is non-negative guarantees that $\mathrm{Re}(\phi, -\mathrm{i}H\phi) \leqslant 0$, for any state ϕ. The operator $-\mathrm{i}H$ is then said to be dissipative (Exner 1985, p 152) and the Trotter product formula holds for the one-parameter semi-group of time evolution associated with $-\mathrm{i}H$. The derivation of the Feynman integral as a limit of multiple ordinary integrals can therefore be carried through (Exner 1985, ch 6) with the following results (Wright 1984) for the propagator $K(e, 0)$ between the points $\boldsymbol{x}_0$ at time t_0 and $\boldsymbol{x}_e$ at time t_e:

$$K(e, 0) = \lim_{n\to\infty} K_n(e, 0), \tag{2.2}$$

$$K_n(e, 0) = A_n \int \mathrm{d}\boldsymbol{x}_1 \ldots \int \mathrm{d}\boldsymbol{x}_{n-1} \exp\left(\frac{\mathrm{i}S_n(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n)}{\hbar}\right)$$

where $\mathrm{d}\boldsymbol{x}_j = (\mathrm{d}x_1)_j \ldots (\mathrm{d}x_k)_j$, and the action is given by

$$S_n(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n) = \Delta t \sum_{j=1}^{n} \left(\frac{(\boldsymbol{x}_j - \boldsymbol{x}_{j-1})^2}{2\Delta t^2} - V(\boldsymbol{x}_{j-1})\right). \tag{2.3}$$

In this equation the quadratic term represents a k-dimensional dot product, the normalisation constant $A_n = (2\pi\mathrm{i}\hbar\Delta t)^{-nk/2}$ and $\Delta t = (t_e - t_0)/n$. The individual integrals $\mathrm{d}\boldsymbol{x}_j$ are carried out over the real configuration space $\boldsymbol{R}^k$ of the system. The analyticity assumptions made above show that the integrand is analytic except for the fixed singularities of the potential; therefore, for each j, i.e. for each time step, the integral over each of the k coordinates $(x_1, \ldots, x_k)$ can be distorted away from the real axis $(-\infty, \infty)$ to any desired contour as long as none of the fixed singularities are included. Note that the endpoints of the contours also remain fixed so that each individual contour looks like the example in figure 1. Since the contour avoids the

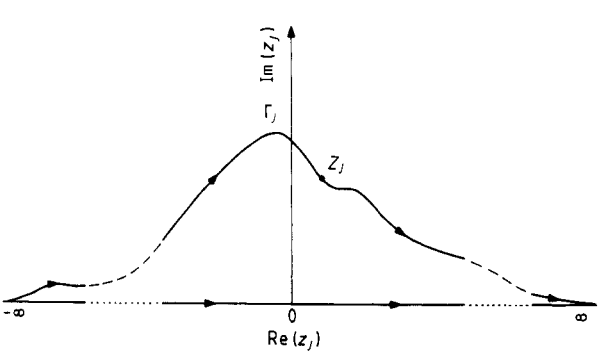

Figure 1. The contour Γ_j at the jth time step is asymptotically pinched to the real axis and passes through the point Z_j determined by the solution to the complex classical equations of motion.

fixed singularities, convergence problems can only arise from the behaviour of the integrand at infinity, where the kinetic energy term is dominant. Here the fact that the contours are asymptotically pinched to the real axis is essential. The explicit expression (2.3) for the action S_n shows that the convergence of the integral over x_j is equivalent to the convergence of integrals of the form

$$I = \int_{-\infty}^{\infty} dx \exp(ix^2 + wx)$$

where $w \propto (x_{j+1} + x_{j-1})$ is in general complex, since the neighbouring variables will not lie on the asymptotic parts of their contours. This integral is of the Fresnel form and converges for any complex w. Thus the convergence of the new representation is guaranteed by the same mechanism, i.e. rapid oscillation, that controls the convergence of the original real representation (2.2). The new representation is

$$K_n(e, 0) = A_n \int_{\Gamma(1)} dz_1 \ldots \int_{\Gamma(n-1)} dz_{n-1} \exp\left(\frac{iS_n(z_0, \ldots, z_n)}{\hbar}\right) \tag{2.4}$$

where $\boldsymbol{\Gamma}(j) = (\Gamma_1(j), \ldots, \Gamma_k(j))$, is a k-tuple composed of the integration contours $\Gamma_\alpha(j)$ in the αth coordinate plane at the jth time step and $dz_j = (dz_1)_j \ldots (dz_k)_j$. For each (j, α) the contour $\Gamma_\alpha(j)$ is asymptotically pinched to the real axis and excludes any singularities of the action $S(z_1, \ldots, z_\alpha, \ldots, z_k)$ in the variable z_α; otherwise the contour can be arbitrarily chosen. The action $S_n(z)$ is the analytic continuation of (2.3). For the class of analytic potentials with fixed singularities we now have a new representation, equivalent to the original one, of the discretised form of the Feynman integral. The final result for the propagator $K(e, 0)$ is obtained by taking the $n \to \infty$ limit of (2.4).

In the usual case of real-path Feynman integrals it is customary to describe the $n \to \infty$ limit in terms of a functional 'integral' over the class of continuous paths. It is of some interest to construct the analogous continuous description of the complex-path integral. To this end we regard the assignment $t_j \to \boldsymbol{\Gamma}(j)$ as a function from the discrete times t_j to the set $\boldsymbol{I}^k$ of k-tuples of asymptotically pinched contours. We want to extend this to a function $\boldsymbol{\Gamma}(t)$ which is continuous in t. For this purpose it is not sufficient merely to replace the discrete time t_j by the continuous time t; we must also introduce a notion of distance in $\boldsymbol{I}$ in order to define $\boldsymbol{\Gamma}(t)$ as a continuous function. One suitable distance function $d(\boldsymbol{\Gamma}, \boldsymbol{\Gamma}')$ is given by

$$d(\boldsymbol{\Gamma}, \boldsymbol{\Gamma}') = \sup\{|\mathrm{Im}(z_\alpha - z'_\alpha)|: z_\alpha \in \Gamma_\alpha, z'_\alpha \in \Gamma'_\alpha \ \mathrm{Re}(z_\alpha - z'_\alpha) = 0, \alpha = 1, \ldots, k\}$$

i.e. the largest vertical separation of any two contours. As $n \to \infty$, $t_j \to t$, and continuity of $\boldsymbol{\Gamma}(t)$ simply means that

$$d(\boldsymbol{\Gamma}(t + \Delta t), \boldsymbol{\Gamma}(t)) \to 0 \qquad \text{as } \Delta t \to 0.$$

The choice of a function $\boldsymbol{\Gamma}(t)$ is analogous to the choice of an ordinary integration contour, and it defines our notion of functional contour integration. Just as for the normal contour integral, it is essential to choose the functional contour to exclude singularities of the integrand. For a given functional contour $\boldsymbol{\Gamma}(t)$, we define a $\boldsymbol{\Gamma}$ path to be a continuous function $z(t) = (z_1, \ldots, z_k)$ which satisfies $z_\alpha(t) \in \Gamma_\alpha(t)$ for all t. The continuity of $\boldsymbol{\Gamma}(t)$ is necessary for the consistency of this definition. The $n \to \infty$ limit of (2.4) is then formally written as a continuous functional integral:

$$K(e, 0) = \int_{\boldsymbol{\Gamma}} \boldsymbol{D}(z(\cdot)) \exp\left(\frac{iS(z(\cdot))}{\hbar}\right) \tag{2.5}$$

where

$$S(z(\cdot)) = \int_{t_0}^{t_e} dt \left[\frac{1}{2} \left(\frac{dz}{dt} \right)^2 - V(z) \right] \tag{2.6}$$

and $\boldsymbol{F}$ is the set of all Γ paths joining the initial and final points x_0 and x_e. Equations (2.5) and (2.6) are the main results of this paper.

The notion of Γ paths can be given a geometrical interpretation as follows. Consider the set $\boldsymbol{M}(\Gamma) = \{(t, w_1, \ldots, w_k): t \in \boldsymbol{R},\ w_\alpha \in \Gamma_\alpha(t)\}$. This set can be regarded as a submanifold of the space $\boldsymbol{M} = \{(t, u_1, \ldots, u_k): t \in \boldsymbol{R},\ u_\alpha \in C,\ \alpha = 1, \ldots, k\}$. The (real) dimension of $\boldsymbol{M}$ is $2k+1$, whereas the dimension of the submanifold $\boldsymbol{M}(\Gamma)$ is $k+1$. This formulation has the virtue of being visualisable for the case $k=1$, where $\boldsymbol{M}$ is three dimensional and $\boldsymbol{M}(\Gamma)$ is two dimensional; this case is shown in figure 2. In this setting a Γ path is simply a curve in $\boldsymbol{M}$ which is required to lie in the submanifold $\boldsymbol{M}(\Gamma)$. By contrast, the naive generalisation to the complex case would involve sums over all paths in $\boldsymbol{M}$. On some of these paths the kinetic energy term in (2.6) would assume large positive imaginary values. This would produce Gaussian divergences in the path integral. A glance at figure 2, shows that the restriction to Γ paths prevents this divergence. Since the allowed paths must pass through the given contours, the asymptotic pinching condition guarantees that large velocities can only occur in an essentially real direction. This means that large kinetic energies are also essentially real and the convergence argument given in § 2 applies. Thus the functional contour method, which we have derived from the real-path integral, supplies the prescription required to make sense of the use of complex Feynman paths.

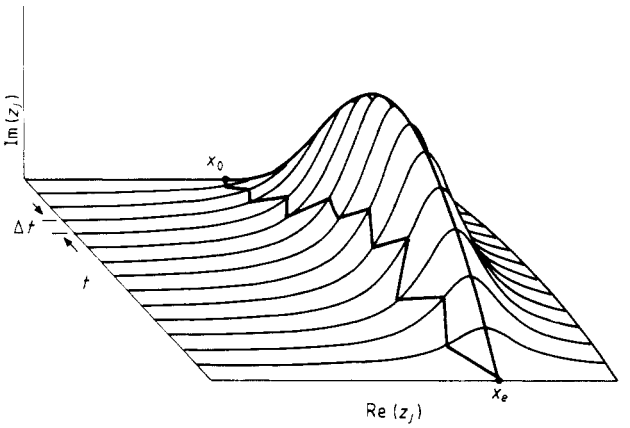

Figure 2. The submanifold $\boldsymbol{M}(\Gamma)$ defining the functional contour integral for a one-dimensional problem. The surface $\boldsymbol{M}(\Gamma)$ is shown ruled by contours at successive time steps. The smooth curve through the peaks represents the complex classical trajectory and the broken line path represents a typical approximate Feynman path.

3. WKB calculations

The purpose of this section is merely to show how the functional contour method allows one to obtain WKB results for absorptive potentials in a simple manner. The calculations will be carried out in the usual approximation in which only second-order terms in the variation of the action are retained; consequently we will simplify the

discussion by restricting ourselves to one-dimensional systems. The generalisation of the argument to higher dimensions will be obvious. Since the use of complex Feynman paths is unfamiliar, we will carry out the calculation in the discretised form and then take the limit. This allows the details of the choice of contours to be explicitly displayed.

The absorptive term in the potential will in general prevent the existence of a real trajectory extremalising the action; however, we can look for a complex solution of the extremal condition

$$\partial S_n(z_0, \ldots, z_n)/\partial z_j = 0 \qquad j = 1, \ldots, n-1$$

with the end conditions $z_0 = x_0$ and $z_n = x_e$. If there is more than one solution, there will be several terms in the WKB approximation for the propagator. For simplicity, we assume that the solution is unique. Denote the solution by Z_j, then the expression (2.3) for S_n yields the difference equations

$$(Z_{j+1} - 2Z_j - Z_{j-1})/\Delta t^2 = -(\partial V(z)/\partial z)|z = Z_j$$

which form the discrete approximation to the analytically continued equation of motion:

$$\mathrm{d}^2 Z/\mathrm{d}t^2 = -\partial V(Z)/\partial Z.$$

We choose the functional contour $\Gamma(t)$ to pass through $Z(t)$, i.e. in the discrete approximation Γ_j passes through Z_j for each j. The stationary point is expected to dominate the integral so the action can be expanded about the solution Z_j. The algebra is the same as for the real case, so the result is

$$K_n(e, 0) = A_n F_n \exp[\mathrm{i} S_n(Z)/\hbar]$$

$$F_n = \int_{\Gamma_1} \mathrm{d}z_1 \ldots \int_{\Gamma_{n-1}} \mathrm{d}z_{n-1} \exp\left(\frac{\mathrm{i}\Delta t \zeta^{\mathrm{T}} \sigma \zeta}{\hbar}\right) \tag{3.1}$$

where ζ is a column matrix with elements $\zeta_j = (z_j - Z_j)$. The contour Γ_j passes through the origin in the ζ_j plane and the matrix σ is given by (Schulman 1981)

$$\sigma = (1/2\Delta t^2)J - \tfrac{1}{2}W$$

where W is

$$W_{ij} = \delta_{ij}(\partial^2 V/\partial z^2)|z = Z_j$$

and $J_{ii} = 2$, $i = 1, \ldots, n-1$; $J_{i,i+1} = -1$, $i = 1, \ldots, n-2$; $J_{i-1,1} = -1$, $i = 2, \ldots, n-1$; all other $J_{ij} = 0$. As usual we make no attempt to determine the corrections due to higher-order terms in ζ; however, it should be noted that in this connection the complex paths introduce no new problems as opposed to the real-path calculation. This follows from the restriction to asymptotically pinched contours.

The standard method for evaluating Gaussian integrals like (3.1) involves expanding the integration variable ζ in the eigenvectors of the matrix σ (Schulman 1981) and this can be a problem in the complex case. When σ is real-symmetric the existence of a complete set of eigenvectors is assured, but this is not guaranteed when σ is complex-symmetric. An alternative sufficient condition for completeness is that all the eigenvalues of a σ be distinct (Lancaster 1962). If there is degeneracy in the eigenvalues, we can imagine removing it by a small alteration in the potential. The calculation with the altered potential could be carried out by the standard method and the original integral recovered by evaluating this function of the potential at the original value.

This will normally give the correct result if there is no singularity, i.e. as long as the original integral converges. Since the integral is known to converge, it is reasonable to limit our attention to matrices σ having a complete set of eigenvectors.

Let $u_1, \ldots, u_n$ be the complete set of eigenvectors, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, then set

$$\zeta = \sum_j \alpha_j u_j$$

and use the coefficients α_j as new integration variables. The integal now has the form

$$F_n = (\Delta t)^{-n/2} \int \mathrm{d}\alpha_1 \ldots \int \mathrm{d}\alpha_n \exp\left(\frac{\mathrm{i}}{\hbar} \sum_j \lambda_j \alpha_j^2\right).$$

The specification of the full integration domain in terms of the variables α would be very complicated, but we are only interested in a small patch near the origin. The calculation from this point on is a standard steepest-descents problem (Morse and Feshbach 1953). The contours Γ_j are chosen so that near the origin $\alpha_j = s_j \exp(\mathrm{i}\phi_j)$, where s_j is real and

$$\phi_j = \frac{\pi}{4} - \frac{\arg(\lambda_j)}{2}.$$

This yields

$$K_n = \left[\left(\frac{m}{2\pi \mathrm{i}\hbar}\right)^{(n+1)/2} (\Delta t)^{-n-1/2} \prod_j \left(\mathrm{i}\pi \frac{\hbar}{\lambda_j}\right)^{1/2}\right] \exp\left(\frac{\mathrm{i}S_n}{\hbar}\right)$$

which has the same form as the standard result for the real-potential case (Schulman 1981). The fact that the eigenvalues λ_j are complex has no effect on the subsequent calculations, so the $n \to \infty$ limit is given by (Schulman 1981)

$$K(e, 0) = (2\pi \mathrm{i}\hbar f(t_e))^{1/2} \exp\left(\frac{\mathrm{i}S(z(\cdot))}{\hbar}\right)$$

where $f(t)$ is the Jacobi field which is defined by

$$\mathrm{d}^2 f / \mathrm{d}t^2 + W(t) f(t) = 0$$

$$W(t) = (\partial^2 V(z)/\partial z^2)|z = Z(t)$$

$$f(t_0) = 0 \qquad f(t_e) = 1.$$

Thus the expected WKB limit of the propagator for an absorptive potential is obtained from the functional contour representation of the path integral by determining the path of stationary complex action. An application to the simple case of an absorptive harmonic oscillator potential was given previously (Garrison and Wright 1985).

4. Spherically symmetric potentials

We now consider a three-dimensional problem described by a complex spherically symmetric potential. The function $V(\boldsymbol{x})$ then depends on $\boldsymbol{x}$ only through the combination $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. Analytic continuation in the cartesian coordinates cannot produce a potential with fixed singularities, since the function $r(\boldsymbol{x})$ has a branch point in x_1, say, which depends on the values of the other coordinates. Fortunately, this

problem is eliminated by the formulation of the real-path integral representation in terms of radial path integrals (Edwards and Gulayev 1964, Peak and Inomata 1969, Langguth and Inomata 1979). In this approach one writes the propagator in a partial wave expansion:

$$K(e,0)=\sum_{l=0}^{\infty}\frac{2l+1}{4\pi r_e r_0}K^l(r_e,r_0)P_l(\cos\theta)$$

$$\cos\theta=\cos\theta_e\cos\theta_0+\sin\theta_e\sin\theta_0\cos(\phi_e-\phi_0).$$

The coefficient K^l is given by the following discretised path integral:

$$K^l=\lim_{n\to\infty}K_n^l$$

$$K_n^l=A_n\int_0^{\infty}\mathrm{d}r_1\ldots\int_0^{\infty}\mathrm{d}r_{n-1}\exp\left(\frac{\mathrm{i}S_n}{\hbar}\right)$$

$$S_n=\Delta t\sum_j\left(\frac{(r_j-r_{j-1})^2}{2\Delta t^2}-\frac{l(l+1)}{2r_jr_{j-1}}-V(r_j)\right).$$

We now assume that $V(r)$ is analytic in r, with possible isolated singularities. The integrals over $r_1,\ldots,r_{n-1}$ can then be deformed at will, provided that singularities of $V(r)$ are avoided and that each contour is pinched to the real axis at $r=\infty$ and $r=0$ and also remains in the right half of the complex r plane. The last restriction is necessary in order to guarantee the convergence of each integral at $r_j=0$, regardless of the values of the other variables. We thus get a representation similar to (2.4) with $k=1$ and contours as just specified. The resemblance to the one-dimensional problem can be enhanced by first making a Langer transformation in which r is replaced by $w=\ln(r)$. The origin is then removed to $-\infty$ and the general formula (2.4) can be used after rewriting the action as a function of w.

5. Tunnelling

It has often been suggested that tunnelling through a real potential barrier could be treated by the use of complex classical trajectories. On the other hand, a successful path integral treatment of tunnelling was given by McLaughlin (1972b) which involves real paths but complex time. We would like to point out that complex time is *required* in one-dimensional tunnelling calculations. Consider a real potential with a barrier which lies between the points x_0 and x_1. Suppose that there is a complex trajectory $z(t)$ joining these two points which corresponds to an energy E lying below the potential maximum, i.e.

$$\mathrm{d}z/\mathrm{d}t=[2(E-V(z))]^{1/2}\qquad z(0)=x_0\qquad z(T)=x_1. \tag{5.1}$$

Let Γ be the curve in the z plane described by $z(t)$. Then the time interval T can be evaluated by inverting (5.1):

$$T(\Gamma)=\int_{\Gamma}\frac{\mathrm{d}z}{[2(E-V(z))]^{1/2}}.$$

Since the potential is real there is a second solution given by $z^*(t)$, so that $T^*(\Gamma) = T(\Gamma^*)$. Consequently the imaginary part of T is given by the difference between the two integrals, i.e.

$$\text{Im } T = \int_\gamma \frac{dz}{[2(E - V(z))]^{1/2}}$$

for any contour γ enclosing the branch points of the integrand defined by $E = V(z)$. Thus Im T cannot vanish and a complex time is required to treat the problem of tunnelling. In McLaughlin's work a contour in complex time is chosen so that the trajectory remains real. In higher-dimensional problems it will not be possible in general to keep all the coordinates real. In this context the use of functional contour integrals may be advantageous, since the contours can be deformed to follow the behaviour of the variables required in the tunnelling problem.

6. Summary and conclusions

We have shown that complex paths can be consistently introduced into Feynman integrals by restricting the paths to a certain submanifold in the complex configuration space of the problem. This notion of functional contour integration produces a representation of the propagator which is free of the divergences associated with proposals to define the path integral over all complex paths. For a non-relativistic k-dimensional system, we must assume that the singularities in the analytically continued potential are fixed, i.e. the location of a singularity in one of the variables is independent of the singularities in the other variables. This condition is automatically satisfied for one-dimensional problems, but it does impose restrictions on the potential for the general ($k > 1$) case. As an example of an admissible potential for a multi-dimensional problem one could consider a molecular model in which the interaction term is a polynomial in the coordinates $x_1, \ldots, x_k$. Since the remainder of the potential is a sum of terms, each involving only one coordinate, the fixed singularity condition is satisfied. For the important special case of a spherically symmetric potential in three dimensions, the fixed singularity condition can be avoided by first expressing the propagator as a partial wave expansion and then applying the functional contour method to the remaining (one-dimensional) radial problem. Since one-dimensional and spherically symmetric problems comprise the bulk of the interesting applications, it would seem that the fixed singularity condition is not a serious limitation in practice.

The most obvious advantage of the complex path formulation is that the WKB results for absorptive potentials can be obtained by a straightforward stationary phase calculation as seen in § 3. The utility of this formulation for tunnelling problems is not clear, since the one-dimensional problem requires the use of complex time and the trajectory can be kept real. However, it is possible that complex paths can be used in the description of tunnelling processes in higher dimensions.

Finally we should briefly indicate the relation, or rather the lack of relation, between functional contour integration, Wick rotation and path integral representations of the partition function. Both of the latter notions are loosely related to the ideas of McLaughlin (1972b) discussed in § 5. In relativistic field theories it is sometimes useful to consider a rotation of the time (or energy) coordinate into the imaginary axis, in order to obtain a Euclidean form of the theory. This is similar to McLaughlin's idea of using a complex time variable while keeping the spatial coordinates real. By contrast

we propose to keep the time coordinate real but allow the spatial coordinates to acquire imaginary parts in a way strictly limited by the use of Γ paths. In statistical mechanics it is also often useful to calculate the partition function as a Wiener integral, which may be thought of as an analytic continuation of the Feynman propagator to purely imaginary time. Again this has no bearing on our proposal, which is intended for application to scattering and propagation problems.

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